Stability Analysis in Geomechanics by Linear Programming. I: Formulation

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Abstract: The limit analysis of stability problems in geomechanics is formulated as a pair of primal-dual linear programs that encode, respectively, the kinematic and static limit theorems in a discrete version. The failure surface can take any arbitrary shape. The soil domain is divided into rigid elements connected by interfacing Mohr-Coulomb layers. For an assumed finite element mesh, the solution of either linear program identifies the critical collapse mechanism among all the possible failure mechanisms contained within the mesh, and gives the associated values of both the static and kinematic variables as well as the critical load parameter. This solution is both kinematically and statically admissible for the discretized system; for the continuum, it is an upper-bound solution. The proposed method is able to deal with external forces acting on a soil mass with varying pore-water pressure, and inhomogeneous materials having both cohesion and internal friction. An illustrative example is presented; this kinematic formulation accurately gives the upper-bound solution.

Introduction

For most geomechanical problems involving stability of slopes, bearing capacity of foundations, or earth pressures on retaining walls, engineers are primarily concerned with the strength of the soil mass at the collapse stage. In the theory of plasticity [see e.g. Chakrabarty (1987)], the assumption of material rigid-perfect plasticity with an associated (or normal) flow rule allows the calculation of a unique collapse load, which may be bracketed by the use of two limit theorems even if it cannot be determined exactly. The static and kinematic theorems of limit analysis have long been exploited for acquiring lower and upper bounds to the collapse load in geotechnical engineering problems (Coulomb 1776; Rankine 1857; *Proc.* 1973; Chen 1975; Salencon 1977). Exact solutions to such problems have not yet been obtained, except for those for simple bearing-capacity problems. Even though it is more realistic to assume a nonassociated flow rule (Drucker 1954; Palmer 1966), solutions adopting the normality assumption have practical values, because, in many stability problems, the kinematic boundary conditions may be insufficiently restrictive for the nature of the flow rule to exert an important influence on the collapse load (Davis and Booker 1973; Zienkiewicz et al. 1975).

Compared with the static approach, the kinematic approach is easier to use. The most widely used class of methods in engineering practice—the limit equilibrium methods (LEM)—are only approximately kinematic methods since most of them assume kinematically inadmissible failure mechanisms and/or violate the yield criterion and the no-tension condition. The violation of static admissibility is usually minor, and many of the methods give good estimates for normal slope stability problems [see e.g. Skempton

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and Hutchinson (1969)]. However, when kinematic admissibility is not rigorously satisfied, the computed collapse load is not necessarily an upper bound (Brinch Hansen 1966). Furthermore, in some unusual circumstances, there are computational difficulties associated with many of the LEM (Whitman and Bailey 1967; Ching and Fredlund 1983). Thus, there is a need to develop an efficient kinematic formulation that satisfies both the kinematic and the static admissibility for a discretized soil mass.

For limit analysis, the theory of linear programming provides not only an efficient computational technique but also a convenient conceptual framework. The two limit theorems can be identified as the dual aspects of a unique LP problem. Although this mathematical formalism has been successfully applied to a wide variety of problems in limit analysis and synthesis of structural systems (*Engineering* 1979; Maier and Munro 1982; Maier and Lloyd Smith 1986), publications on its applications to stability problems in geomechanics are rather few.

The first contribution was by Lysmer (1970), who proposed a static approach for plane-strain problems. Each nonlinear yield criterion was approximated by a number of linear constraints; but an increase in this number considerably increases the computational effort. As a result, the problem was solved by an iterative sequence of linear programs (LPs) with fewer constraints (Lysmer 1970). The problem was later reformulated by Zavelani-Rossi et al. (1974) to improve the computational efficiency.

The linearization of the yield criterion was discarded by some authors, who opt for nonlinear programming (Frémond and Salencon 1973; Basudhar et al. 1979). Nonetheless, the nonlinear programming approach lacks the important feature of the linear programming formulation—the duality that relates both conceptually and computationally the static and kinematic aspects of the problem.

A mixed static-kinematic approach was adopted by Casciaro and Cascini (1982), who pursued an approximation to the collapse load rather than a proper bound. However, the stress and velocity fields, while admissible independently, together may become inadmissible, and spurious oscillations in the solution for stresses and/or velocities may consequently result (Casciaro and Cascini 1982).

The foregoing drawbacks associated with the use of the nonlinear yield criterion and the mixed interpolation do not exist in a kinematic approach that uses rigid elements together with a linear yield criterion. The formulation to be considered in this paper was suggested by Munro (1982) as an adaptation of an earlier linear programming model for yield-line limit analysis of thin plates in bending (Da Fonseca et al. 1977). With plasticity confined to Mohr-Coulomb layers at the edges of rigid elements, it was shown that for the fundamental relations to lead to a primal-dual pair of LPs, it was necessary, inter alia, to have a complementarity condition corresponding to the imposition of the associated flow rule. However, in this formulation, the static and kinematic admissibility necessary for the linear programming formulation was not achieved, because insufficient degrees of freedom were assigned to each side of the finite element. Furthermore, one of the suggested ways of evaluating the safety factor is inappropriate for frictional materials, and the presence of pore-water pressure was not considered.

In the formulation by Martins et al. (1981) using a similar approach, each element appears to be connected to its neighbors by n contact points; but the authors did not specify the conditions sufficient to achieve the relevant static and kinematic admissibility. It is proposed in the present paper that three degrees of freedom on each element side are necessary and sufficient to achieve the static equilibrium and kinematic compatibility of a rigid element.

The purpose of this paper is to develop a kinematic formulation that satisfies both the static and kinematic admissibility of a discretized soil mass with varying pore-water pressure, for inhomogeneous materials having both cohesion and internal friction. The present paper introduces the necessary adjustments to Munro's (1982) formulation. A simple example is presented to illustrate the procedure of using the revised formulation. For the brevity of this paper, further example problems and investigation of the computational capacity of this new formulation are presented in a companion paper (Chuang 1992). Subsequent to the writing of this paper and its companion, the writer was made aware of a later publication by Martins (1982) that contains a similar formulation. It appears to have the last two deficiencies noted in connection with (Munro 1982), and it does not indicate explicitly the conditions for static and kinematic admissibility.

Linear Programming Formulation

The soil mass is conceived as a continuum under plane-strain conditions. The domain of interest is divided into rigid elements of triangular or quadrilateral shapes, as shown in Fig. 1. These elements are connected at their interfaces through thin layers of a rigid–perfectly plastic material that obeys the Mohr-Coulomb yield criterion.

The nodes of the elements, the elements, and the connecting layers are numbered in some arbitrary manner (Fig. 1). The formulation for triangular elements is presented in this section; the extension to quadrilateral elements is straightforward. In Fig. 2, the numbers i, j, k are nodal numbers in clockwise direction around the element. For element side i, which is opposite to node i, Δs_i denotes its length, θ_i denotes its clockwise rotation from the horizontal, and H_i denotes the height of the triangle measured perpendicular to this side, as indicated in Fig. 2.

The normal stresses and strains between each Mohr-Coulomb layer and the connected element edge are concentrated at two contact nodes, each located at one-third of the layer length (see Figs. 3 and 4). In this paper, a contact node additional to that in (Munro 1982) is introduced for each element edge in order to ensure static equilibrium and kinematic compatibility of the rigid element. If the normal stress distribution is assumed to be linear and the normal forces at the two contact nodes are nontensile, then nowhere along the layer will the contact stresses be tensile, as required by the material properties.

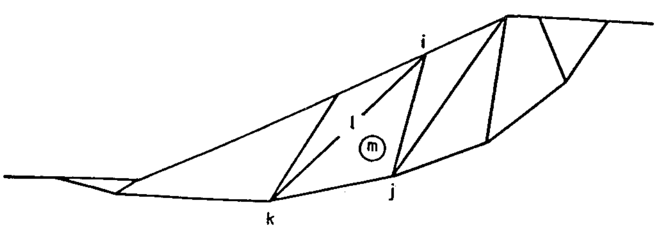


FIG. 1. Discretization of Plane Section into Rigid Elements

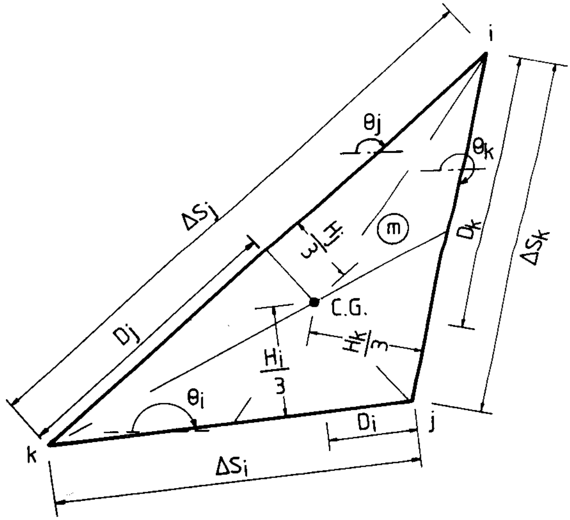


FIG. 2. Geometry of Element *m*

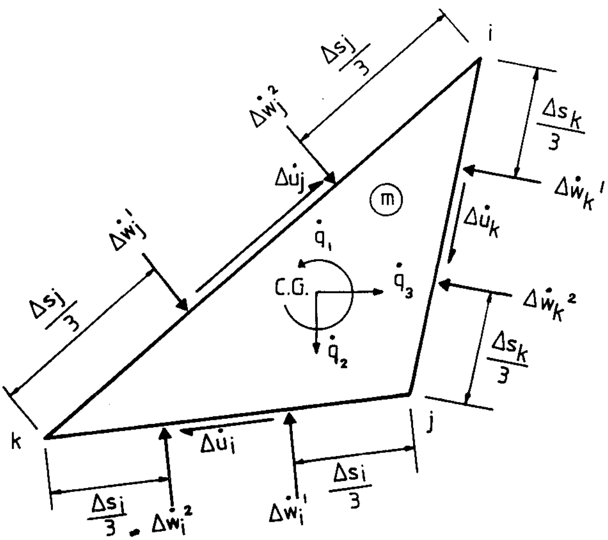


FIG. 3. Kinematics of Element *m*

Kinematics

For the rigid element (see Fig. 3), the displacement rates ($\mathbf{\Delta\dot{u}}$, $\mathbf{\Delta\dot{w}}$) of the element sides can be expressed in terms of the three degrees of freedom ($\dot{\mathbf{q}}$) of the centroid of the element, as given in (1*a*)

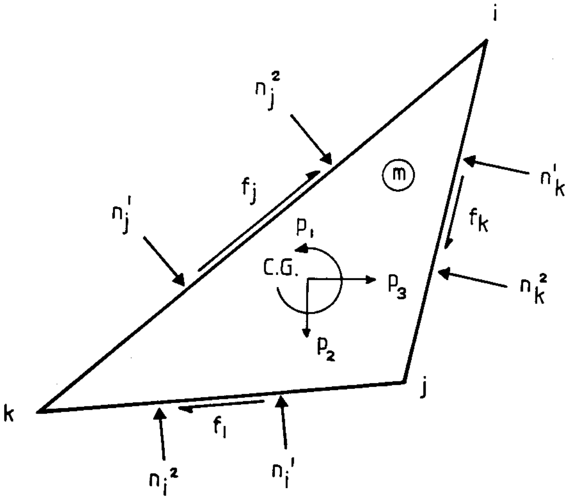


FIG. 4. Statics of Element m

$$\begin{bmatrix} \Delta\dot{u}_i \\ \Delta\dot{u}_j \\ \Delta\dot{u}_k \\ \hline \Delta\dot{w}_i^1 \\ \Delta\dot{w}_i^2 \\ \Delta\dot{w}_j^1 \\ \Delta\dot{w}_j^2 \\ \Delta\dot{w}_k^1 \\ \Delta\dot{w}_k^2 \end{bmatrix} = \begin{bmatrix} \frac{-H_i}{3} & -\sin\theta_i & -\cos\theta_i \\ \frac{-H_j}{3} & -\sin\theta_j & -\cos\theta_j \\ \frac{-H_k}{3} & -\sin\theta_k & -\cos\theta_k \\ \hline D_i - \frac{\Delta s_i}{3} & -\cos\theta_i & \sin\theta_i \\ -\left(\frac{2\Delta s_i}{3} - D_i\right) & -\cos\theta_i & \sin\theta_i \\ -D_j - \frac{\Delta s_j}{3} & -\cos\theta_j & \sin\theta_j \\ -\left(\frac{2\Delta s_j}{3} - D_j\right) & -\cos\theta_j & \sin\theta_j \\ -D_k - \frac{\Delta s_k}{3} & -\cos\theta_k & \sin\theta_k \\ -\left(\frac{2\Delta s_k}{3} - D_k\right) & -\cos\theta_k & \sin\theta_k \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \quad \text{(1a)}$$

where $\Delta\dot{u}_i$ = tangential velocity; and $\Delta\dot{w}_i^1$ and $\Delta\dot{w}_i^2$ = normal velocities at the two contact nodes along side i. Generally

$$\begin{bmatrix} \Delta\dot{\mathbf{u}} \\ \text{----} \\ \Delta\dot{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^m \\ \text{----} \\ \mathbf{A}_2^m \end{bmatrix} \dot{\mathbf{q}} \qquad \text{(1}b\text{)}$$

where $\mathbf{A}_1^m$ and $\mathbf{A}_2^m$ = matrices of purely geometric properties of the element.

Statics

The equipollent tangential (**f**) and normal (**n**) forces transmitted from the element onto the three Mohr-Coulomb layers connected to its edges are related to the body forces (**p**), which are applied at the centroid (see Fig. 4), as given in (2*a*).

$$
\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} =
\left[\begin{array}{ccc|cccccc}
\frac{-H_i}{3} & \frac{-H_j}{3} & \frac{-H_k}{3} & D_i - \frac{\Delta s_i}{3} & -\left(\frac{2\Delta s_i}{3} - D_i\right) & D_j - \frac{\Delta s_j}{3} & -\left(\frac{2\Delta s_j}{3} - D_j\right) & D_k - \frac{\Delta s_k}{3} & -\left(\frac{2\Delta s_k}{3} - D_k\right) \\
-\sin\theta_i & -\sin\theta_j & -\sin\theta_k & -\cos\theta_i & -\cos\theta_i & -\cos\theta_j & -\cos\theta_j & -\cos\theta_k & -\cos\theta_k \\
-\cos\theta_i & -\cos\theta_j & -\cos\theta_k & \sin\theta_i & \sin\theta_i & \sin\theta_j & \sin\theta_j & \sin\theta_k & \sin\theta_k
\end{array}\right]
\begin{bmatrix} f_i \\ f_j \\ f_k \\ \hline n_i^1 \\ n_i^2 \\ n_j^1 \\ n_j^2 \\ n_k^1 \\ n_k^2 \end{bmatrix} \quad \ldots\ldots\ldots\ldots (2a)
$$

Generally

$$\left[(\mathbf{A}_1^m)^T \;\middle|\; (\mathbf{A}_2^m)^T \right] \left[\begin{array}{c} \mathbf{f} \\ \hline \mathbf{n} \end{array} \right] = \mathbf{p} \qquad (2b)$$

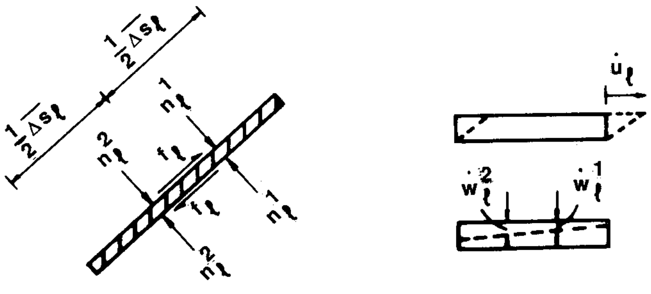
where superscript T = transposed matrix.

The transpose relation linking (1b) and (2b) exhibits the customary static-kinematic duality (SKD) (Munro and Smith 1972).

The sign convention of the forces [Fig. 5(a)] is that normal forces are considered positive when compressive, and tangential forces positive when in clockwise sense.

The rigid elements displace when the interfacing layers deform to produce a plastic collapse mode [Fig. 5(b)]. For compatibility, the deformation rate of a layer, say numbered l, and the displacement rates of the sides of adjacent elements numbered m and n (Fig. 6) have the following relations:

$$\dot{u}_l = -\Delta\dot{u}_m - \Delta\dot{u}_n \qquad (3a)$$



(a) (b)

FIG. 5. (a) Sign Convention of Forces; (b) Deformation Rates on Layer l

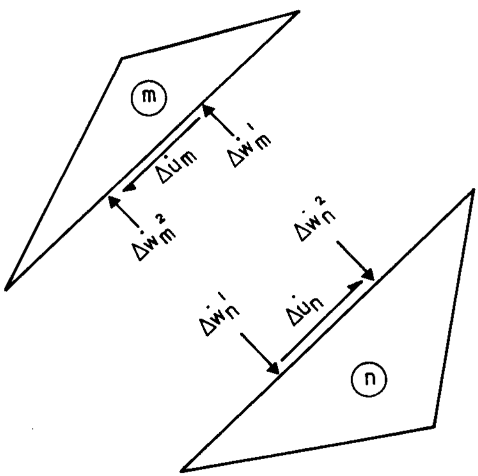


FIG. 6. Displacement Rates of Elements Numbered m and n at Adjacent Sides

$$\dot{w}_l^1 = -\Delta\dot{w}_m^1 - \Delta\dot{w}_n^2 \qquad (3b)$$

$$\dot{w}_l^2 = -\Delta\dot{w}_m^2 - \Delta\dot{w}_n^1 \qquad (3c)$$

The assembly of all such relations in (3) of the system enables the deformation rates ($\dot{\mathbf{u}}$, $\dot{\mathbf{w}}$) of all the layers to be linked to the displacement rates of element sides, and therefore to the generalized displacement rates ($\dot{\mathbf{q}}$)

$$\begin{bmatrix} \mathbf{A}_1 \\ \hline \mathbf{A}_2 \end{bmatrix} \dot{\mathbf{q}} = \begin{bmatrix} \dot{\mathbf{u}} \\ \hline \dot{\mathbf{w}} \end{bmatrix} \qquad (4)$$

Analogously, by considering equilibrium between adjacent elements and the connecting layers, the forces on the Mohr-Coulomb layers (**f**, **n**) can be linked to the element body forces (**p**)

$$\begin{bmatrix} \mathbf{A}_1^T & \mathbf{A}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \hline \mathbf{n} \end{bmatrix} = \mathbf{p} \qquad (5)$$

Again SKD is demonstrated in (4) and (5).

Yield Criterion

The Mohr-Coulomb yield criterion for layer l is

$$|f_l| \le \frac{C_l}{F} + (\zeta_l^1 n_l^1 + \zeta_l^2 n_l^2) - (\zeta_l^1 r_l^1 + \zeta_l^2 r_l^2) \qquad (6)$$

where C_l = integral of the cohesion along layer l. The coefficient ζ_l^i is defined as $\zeta_l^i \equiv (\tan \phi_l'^i)/F$, where $\phi_l'^i$ = angle of internal friction in terms of effective stress at contact node i of layer l, $i = 1, 2$; and F = a reduction factor (which is identical to the conventional safety factor in LEM); r_l^i = integral of the pore pressure along that half of the layer length $\overline{\Delta s_l}$, which is tributary to contact node i, as shown in Fig. 5(a). Let

$$b_l \equiv \frac{C_l}{F} - (\zeta_l^1 r_l^1 + \zeta_l^2 r_l^2) \qquad (7)$$

then inequality (6) can be rearranged as

$$|f_l| - (\zeta_l^1 n_l^1 + \zeta_l^2 n_l^2) \le b_l \qquad (8)$$

For all such layers

$$\mathbf{f} - \mathbf{Z}\mathbf{n} \le \mathbf{b} \qquad (9a)$$

$$-\mathbf{f} - \mathbf{Z}\mathbf{n} \le \mathbf{b} \qquad (9b)$$

where $\mathbf{Z} \equiv \text{diag}[(\zeta_1^1, \zeta_1^2), \ldots, (\zeta_l^1, \zeta_l^2), \ldots, (\zeta_L^1, \zeta_L^2)]$; L = total number of layers; $\mathbf{n} \ge \mathbf{0}$ since no tension is allowed.

In a compact form

$$\begin{bmatrix} \mathbf{I} & -\mathbf{Z} \\ \hline -\mathbf{I} & -\mathbf{Z} \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \hline \mathbf{n} \end{bmatrix} \le \begin{bmatrix} \mathbf{b} \\ \hline \mathbf{b} \end{bmatrix} \qquad (10a)$$

$$\mathbf{n} \ge \mathbf{0} \qquad (10b)$$

where $\mathbf{I}$ = identity matrix.

Let body forces $\mathbf{p}$ consist of two components: fixed loads $\mathbf{p}_f$ and activating forces $\mathbf{p}_a = \lambda \mathbf{p}_o$, where λ = load parameters, and $\mathbf{p}_o$ are the activating forces associated with a unity value of the parameter. Combining (5) and (10), the following static admissibility conditions are obtained.

$$\left[\begin{array}{c|c|c} \cdot & \mathbf{I} & -\mathbf{Z} \\ \hline \cdot & -\mathbf{I} & -\mathbf{Z} \\ \hline \mathbf{p}_o & -\mathbf{A}_1^T & -\mathbf{A}_2^T \end{array}\right] \left[\begin{array}{c} \lambda \\ \hline \mathbf{f} \\ \hline \mathbf{n} \end{array}\right] \begin{array}{c} \leq \\ \leq \\ = \end{array} \left[\begin{array}{c} \mathbf{b} \\ \hline \mathbf{b} \\ \hline -\mathbf{p}_f \end{array}\right] \quad \begin{array}{r} \ldots (11a) \\ \ldots (11b) \\ \ldots (11c) \end{array}$$

$$\mathbf{n} \geq \mathbf{0} \quad \ldots (11d)$$

which give the equalities

$$\left[\begin{array}{c|c|c} \cdot & \mathbf{I} & -\mathbf{Z} \\ \hline \cdot & -\mathbf{I} & -\mathbf{Z} \\ \hline \mathbf{p}_o & -\mathbf{A}_1^T & -\mathbf{A}_2^T \end{array}\right] \left[\begin{array}{c} \lambda \\ \hline \mathbf{f} \\ \hline \mathbf{n} \end{array}\right] + \left[\begin{array}{c} \mathbf{s}^+ \\ \hline \mathbf{s}^- \\ \hline \cdot \end{array}\right] = \left[\begin{array}{c} \mathbf{b} \\ \hline \mathbf{b} \\ \hline -\mathbf{p}_f \end{array}\right] \quad \ldots (12a)$$

$$\mathbf{s}^+ \geq \mathbf{0}; \quad \mathbf{s}^- \geq \mathbf{0}; \quad \mathbf{n} \geq \mathbf{0} \quad \ldots (12b)$$

where $\mathbf{s}^+$ and $\mathbf{s}^-$ = slack variables; and the dot entry $(\cdot)$ indicates a zero vector.

The mechanism deformation rates will be restricted to finite magnitudes through the convenient normalization in (13)

$$\mathbf{p}_o^T \dot{\mathbf{q}} = 1 \quad \ldots (13)$$

Flow Rule

The associated (or normal) flow rule ensures that the relative displacement rate vector of the plasticized layer is normal to the yield locus (Munro 1982). Let the tangential deformation rates ($\dot{\mathbf{u}}$) be expressed in terms of nonnegative components

$$\dot{u}_l = \dot{u}_l^+ - \dot{u}_l^- \quad \ldots (14a)$$

$$\dot{u}_l^+ \geq 0; \quad \dot{u}_l^- \geq 0 \quad \ldots (14b)$$

The flow rule gives, at the two contact nodes of layer l

$$-\dot{w}_l^1 - \zeta_l^1(\dot{u}_l^+ + \dot{u}_l^-) \geq 0 \quad \ldots (15a)$$

$$-\dot{w}_l^2 - \zeta_l^2(\dot{u}_l^+ + \dot{u}_l^-) \geq 0 \quad \ldots (15b)$$

For all such layers

$$-\dot{\mathbf{w}} - \mathbf{Z}^T(\dot{\mathbf{u}}^+ + \dot{\mathbf{u}}^-) \geq \mathbf{0} \quad \ldots (16)$$

Collecting (14) and (16), we have

$$\left[\begin{array}{c|c} \mathbf{I} & -\mathbf{I} \\ \hline -\mathbf{Z}^T & -\mathbf{Z}^T \end{array}\right] \left[\begin{array}{c} \dot{\mathbf{u}}^+ \\ \hline \dot{\mathbf{u}}^- \end{array}\right] \begin{array}{c} = \\ \geq \end{array} \left[\begin{array}{c} \dot{\mathbf{u}} \\ \hline \dot{\mathbf{w}} \end{array}\right] \quad \begin{array}{r} \ldots (17a) \\ \ldots (17b) \end{array}$$

$$\dot{\mathbf{u}}^+ \geq \mathbf{0}; \quad \dot{\mathbf{u}}^- \geq \mathbf{0} \quad \ldots (17c)$$

The transpose relation linking (10) and (17) arises directly from the normality relation connecting the yield criterion with the flow rule.
After substituting (4) into (17), we have

$$\left[\begin{array}{c:c:c} \mathbf{I} & -\mathbf{I} & -\mathbf{A}_1 \\ \hdashline -\mathbf{Z}^T & -\mathbf{Z}^T & -\mathbf{A}_2 \end{array}\right] \left[\begin{array}{c} \dot{\mathbf{u}}^+ \\ \hdashline \dot{\mathbf{u}}^- \\ \hdashline \dot{\mathbf{q}} \end{array}\right] \begin{array}{c} = \\ \geq \end{array} \left[\begin{array}{c} \cdot \\ \hdashline \cdot \end{array}\right] \quad \begin{array}{r} \ldots\ldots (18a) \\ \ldots\ldots (18b) \end{array}$$

$$\dot{\mathbf{u}}^+ \geq \mathbf{0}; \quad \dot{\mathbf{u}}^- \geq \mathbf{0} \quad \ldots\ldots (18c)$$

Assembling (13) and (18), we have the kinematic admissibility conditions

$$\left[\begin{array}{c:c:c} \cdot & \cdot & \mathbf{p}_o^T \\ \hdashline \mathbf{I} & -\mathbf{I} & -\mathbf{A}_1 \\ \hdashline -\mathbf{Z}^T & -\mathbf{Z}^T & -\mathbf{A}_2 \end{array}\right] \left[\begin{array}{c} \dot{\mathbf{u}}^+ \\ \hdashline \dot{\mathbf{u}}^- \\ \hdashline \dot{\mathbf{q}} \end{array}\right] \begin{array}{c} = \\ = \\ \geq \end{array} \left[\begin{array}{c} 1 \\ \hdashline \cdot \\ \hdashline \cdot \end{array}\right] \quad \begin{array}{r} \ldots\ldots (19a) \\ \ldots\ldots (19b) \\ \ldots\ldots (19c) \end{array}$$

$$\dot{\mathbf{u}}^+ \geq \mathbf{0}; \quad \dot{\mathbf{u}}^- \geq \mathbf{0} \quad \ldots\ldots (19d)$$

which give the equalities

$$\left[\begin{array}{c:c:c} \cdot & \cdot & \mathbf{p}_o^T \\ \hdashline \mathbf{I} & -\mathbf{I} & -\mathbf{A}_1 \\ \hdashline -\mathbf{Z}^T & -\mathbf{Z}^T & -\mathbf{A}_2 \end{array}\right] \left[\begin{array}{c} \dot{\mathbf{u}}^+ \\ \hdashline \dot{\mathbf{u}}^- \\ \hdashline \dot{\mathbf{q}} \end{array}\right] - \left[\begin{array}{c} \cdot \\ \hdashline \cdot \\ \hdashline \boldsymbol{\sigma} \end{array}\right] = \left[\begin{array}{c} 1 \\ \hdashline \cdot \\ \hdashline \cdot \end{array}\right] \quad \ldots (20a)$$

$$\dot{\mathbf{u}}^+ \geq \mathbf{0}; \quad \dot{\mathbf{u}}^- \geq \mathbf{0}; \quad \boldsymbol{\sigma} \geq \mathbf{0} \quad \ldots\ldots (20b)$$

where $\boldsymbol{\sigma}$ = surplus variables.
The two sets of conditions (12) and (20) are linked by a complementarity relation (the parity rule), which ensures admissible combinations of static and kinematic solutions. For layer l

$$s_l^+ \dot{u}_l^+ = 0 \quad \ldots\ldots (21a)$$

$$s_l^- \dot{u}_l^- = 0 \quad \ldots\ldots (21b)$$

$$n_l \sigma_l = 0 \quad \ldots\ldots (21c)$$

For all layers, the system parity rule is

$$(\mathbf{s}^+)^T \dot{\mathbf{u}}^+ + (\mathbf{s}^-)^T \dot{\mathbf{u}}^- + \mathbf{n}^T \boldsymbol{\sigma} = 0 \quad \ldots\ldots (22)$$

The complete set of fundamental geomechanical relations for the system at plastic collapse—namely (12), (20), and (22)—can now be assembled in the form of a linear complementarity problem (LCP)

$$\left[\begin{array}{c:c:c} \cdot & \cdot & \mathbf{p}_o^T \\ \hdashline \mathbf{I} & -\mathbf{I} & -\mathbf{A}_1 \\ \hdashline -\mathbf{Z}^T & -\mathbf{Z}^T & -\mathbf{A}_2 \end{array}\right]\left[\begin{array}{c} \dot{\mathbf{u}}^+ \\ \hdashline \dot{\mathbf{u}}^- \\ \hdashline \dot{\mathbf{q}} \end{array}\right] - \left[\begin{array}{c} \cdot \\ \hdashline \cdot \\ \hdashline \boldsymbol{\sigma} \end{array}\right] = \left[\begin{array}{c} 1 \\ \hdashline \cdot \\ \hdashline \cdot \end{array}\right] \quad \ldots (23)$$

$$\left[\begin{array}{c:c:c} \cdot & \mathbf{I} & -\mathbf{Z} \\ \hdashline \cdot & -\mathbf{I} & -\mathbf{Z} \\ \hdashline \mathbf{p}_o & -\mathbf{A}_1^T & -\mathbf{A}_2^T \end{array}\right]\left[\begin{array}{c} \lambda \\ \hdashline \mathbf{f} \\ \hdashline \mathbf{n} \end{array}\right] + \left[\begin{array}{c} \mathbf{s}^+ \\ \hdashline \mathbf{s}^- \\ \hdashline \cdot \end{array}\right] = \left[\begin{array}{c} \mathbf{b} \\ \hdashline \mathbf{b} \\ \hdashline -\mathbf{p}_f \end{array}\right]$$

$$(\mathbf{s}^+)^T\dot{\mathbf{u}}^+ + (\mathbf{s}^-)^T\dot{\mathbf{u}}^- + \mathbf{n}^T\boldsymbol{\sigma} = 0$$

$$\mathbf{s}^+ \geq \mathbf{0}; \quad \mathbf{s}^- \geq \mathbf{0}; \quad \dot{\mathbf{u}}^+ \geq \mathbf{0}; \quad \dot{\mathbf{u}}^- \geq \mathbf{0}; \quad \mathbf{n} \geq \mathbf{0}; \quad \boldsymbol{\sigma} \geq \mathbf{0}$$

The solution of LCP (23), according to Karush-Kuhn-Tucker (KKT) theory (Zangwill 1969), can be obtained through either of the primal-dual LPs, (24) and (25)

$$\text{minimize } Z = [\mathbf{b}^T \mid \mathbf{b}^T \mid -\mathbf{p}_f^T]\left[\begin{array}{c} \dot{\mathbf{u}}^+ \\ \hdashline \dot{\mathbf{u}}^- \\ \hdashline \dot{\mathbf{q}} \end{array}\right]$$

$$\left[\begin{array}{c:c:c} \cdot & \cdot & \mathbf{p}_o^T \\ \hdashline \mathbf{I} & -\mathbf{I} & -\mathbf{A}_1 \\ \hdashline -\mathbf{Z}^T & -\mathbf{Z}^T & -\mathbf{A}_2 \end{array}\right]\left[\begin{array}{c} \dot{\mathbf{u}}^+ \\ \hdashline \dot{\mathbf{u}}^- \\ \hdashline \dot{\mathbf{q}} \end{array}\right] \begin{array}{c} = \\ = \\ \geq \end{array} \left[\begin{array}{c} 1 \\ \hdashline \cdot \\ \hdashline \cdot \end{array}\right] \quad \ldots\ldots\ldots (24)$$

$$\dot{\mathbf{u}}^+ \geq \mathbf{0}; \quad \dot{\mathbf{u}}^- \geq \mathbf{0}$$

(primal linear program). And

$$\text{maximize } W = [1 \mid \cdot \mid \cdot]\left[\begin{array}{c} \lambda \\ \hdashline \mathbf{f} \\ \hdashline \mathbf{n} \end{array}\right]$$

$$\left[\begin{array}{c:c:c} \cdot & \mathbf{I} & -\mathbf{Z} \\ \hdashline \cdot & -\mathbf{I} & -\mathbf{Z} \\ \hdashline \mathbf{p}_o & -\mathbf{A}_1^T & -\mathbf{A}_2^T \end{array}\right]\left[\begin{array}{c} \lambda \\ \hdashline \mathbf{f} \\ \hdashline \mathbf{n} \end{array}\right] \begin{array}{c} \leq \\ \leq \\ = \end{array} \left[\begin{array}{c} \mathbf{b} \\ \hdashline \mathbf{b} \\ \hdashline -\mathbf{p}_f \end{array}\right] \quad \ldots\ldots (25)$$

$$\mathbf{n} \geq 0$$

(dual linear program).

These primal-dual LPs establish, in a discrete version, the kinematic and static limit theorems, respectively, for stability analysis in geomechanics (Drucker and Prager 1952; Drucker et al. 1952). The limit theorems are: (1) Kinematic theorem—the soil mass will collapse if there is any compatible pattern of plastic deformation for which the rate of work of the external loads exceeds the rate of internal dissipation; and (2) static theorem—if an equilibrium distribution of forces can be found that balances the applied load and nowhere violates the yield criterion that includes c' and ϕ', the soil mass will not collapse or will be just at the point of collapse.

For any assumed finite element (FE) mesh, the plastic deformations are confined to the Mohr-Coulomb layers, while the interior of the elements remains undeformed plastically. The optimal solution of the LPs is the complete solution that is both kinematically and statically admissible to the discretized system; it is an upper bound to the complete (or exact) solution of the continuum. These two solutions will be identical provided that the actual collapse mechanism is contained within the FE model. Thus, the accuracy of the solution can be greatly enhanced by a judicious choice of the FE mesh.

EXAMPLE

A very simple example is used to illustrate the assembly of the relevant matrices for LPs (24) and (25), and to validate the linear programming formulation. Fig. 7 shows a vertical cut of a purely cohesive, weighty material. The greatest height that the cut can stand under its own weight without collapsing plastically is to be determined by evaluating the nondimensional parameter $\gamma H/c_u$. If the height H and the undrained cohesion c_u are assumed unity, the problem is reduced to finding the maximum value of the unit weight γ subject to static admissibility conditions.

The domain is discretized into two triangular elements by two potential slip lines. The nodes, layers and elements are numbered as in Fig. 7. To minimize the data for this illustrative example, the correct sense of the generalized displacement rates (as shown in Fig. 8), and of the tangential deformation rates of the layers (i.e. anticlockwise as shown in Fig. 9) is imposed, and this enables one to treat the decision variables $\dot{\mathbf{u}}$, $\mathbf{f}$ and $\dot{\mathbf{q}}$, $\mathbf{p}$ as nonnegative.

Kinematics

Element 1 is attached to only one Mohr-Coulomb layer; hence only one element side need be considered. From the geometry of the element [Fig.

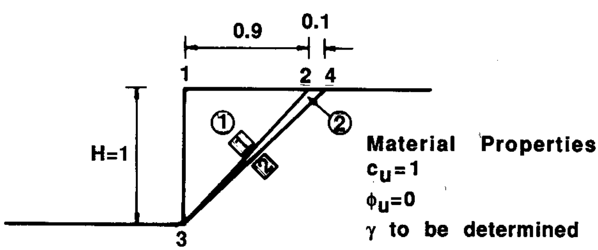


FIG. 7. Description of Problem

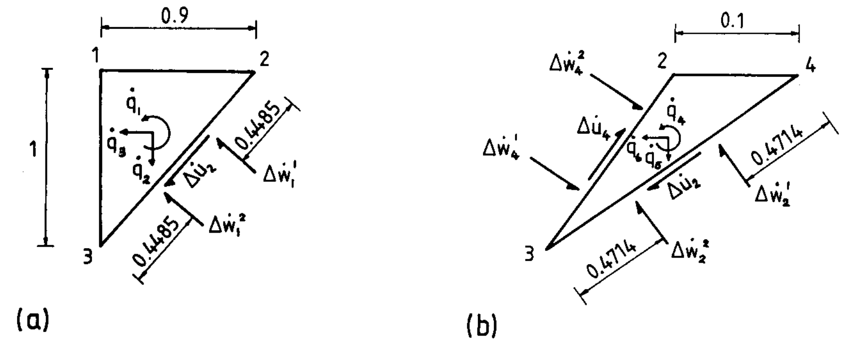


FIG. 8. Kinematics of Elements: (*a*) Element 1; (*b*) Element 2 (not to Scale)

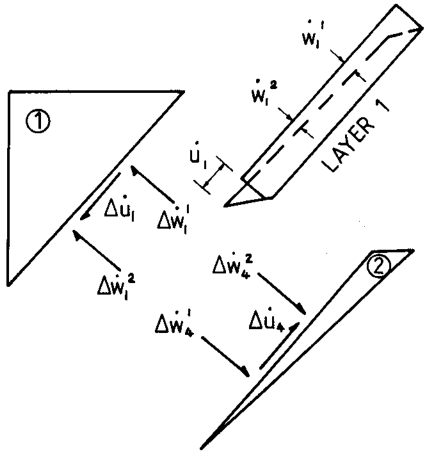


FIG. 9. Relation between Element-Side Displacement Rates and Deformation Rates of Layer 1

8(*a*)], the displacement rates of side 1 can be expressed in terms of the generalized displacement rates of element 1 using (1*a*)

$$\begin{bmatrix} \Delta\dot{u}_1 \\ \Delta\dot{w}_1^1 \\ \Delta\dot{w}_1^2 \end{bmatrix} = \begin{bmatrix} -0.2230 & 0.7433 & 0.6690 \\ 0.2007 & -0.6690 & 0.7433 \\ -0.2478 & -0.6690 & 0.7433 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \quad (26)$$

Similarly, for element 2, which is bounded by two layers [Fig. 8(*b*)]

$$\begin{bmatrix} \Delta\dot{u}_2 \\ \Delta\dot{u}_4 \\ \Delta\dot{w}_2^1 \\ \Delta\dot{w}_2^2 \\ \Delta\dot{w}_4^1 \\ \Delta\dot{w}_4^2 \end{bmatrix} = \begin{bmatrix} -0.0236 & 0.7071 & 0.7071 \\ -0.0248 & -0.7433 & -0.6690 \\ 0.0236 & -0.7071 & 0.7071 \\ -0.4478 & -0.7071 & 0.7071 \\ 0.4708 & 0.6690 & -0.7433 \\ 0.0223 & 0.6690 & -0.7433 \end{bmatrix} \begin{bmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} \quad (27)$$

The deformation rates of layer 1 are related to the displacement rates of side 1 of element 1 and side 4 of element 2 (Fig. 9) in the following way.

$$\dot{u}_1 = \Delta\dot{u}_1 + \Delta\dot{u}_4 \quad \text{(28a)}$$

$$\dot{w}_1^1 = -\Delta\dot{w}_1^1 - \Delta\dot{w}_4^2 \quad \text{(28b)}$$

$$\dot{w}_1^2 = -\Delta\dot{w}_1^2 - \Delta\dot{w}_4^1 \quad \text{(28c)}$$

and for layer 2

$$\dot{u}_2 = \Delta\dot{u}_2 \quad \text{(29a)}$$

$$\dot{w}_2^1 = -\Delta\dot{w}_2^1 \quad \text{(29b)}$$

$$\dot{w}_2^2 = -\Delta\dot{w}_2^2 \quad \text{(29c)}$$

After assembly of relations (28) and (29), the deformation rates ($\dot{\mathbf{u}}$, $\dot{\mathbf{w}}$) of the two layers may be related to the element side displacement rates and hence to the generalized displacement rates ($\dot{\mathbf{q}}$) using (26) and (27)

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \hline \dot{w}_1^1 \\ \dot{w}_1^2 \\ \dot{w}_2^1 \\ \dot{w}_2^2 \end{bmatrix} = \begin{bmatrix} -0.223 & 0.7433 & 0.6690 & -0.0248 & -0.7433 & -0.6690 \\ 0 & 0 & 0 & -0.0236 & 0.7071 & 0.7071 \\ \hline -0.2007 & 0.6690 & -0.7433 & -0.0223 & -0.6690 & 0.7433 \\ +0.2478 & 0.6690 & -0.7433 & -0.4708 & -0.6690 & 0.7433 \\ 0 & 0 & 0 & -0.0236 & 0.7071 & -0.7071 \\ 0 & 0 & 0 & 0.4478 & 0.7071 & -0.7071 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} \quad \text{(30)}$$

Symbolically

$$\begin{bmatrix} \dot{\mathbf{u}} \\ \hline \dot{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \\ \hline \mathbf{A}_2 \end{bmatrix} \dot{\mathbf{q}} \quad \text{(31)}$$

Statics

The forces on the layers can be linked to the body forces of the elements through the system equilibrium equations

$$\mathbf{p} = \begin{bmatrix} \mathbf{A}_1^T & \mathbf{A}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \hline \mathbf{n} \end{bmatrix} \quad \text{(32a)}$$

where

$$\mathbf{p}^T = [p_1\ p_2\ p_3\ p_4\ p_5\ p_6] \quad \text{(32b)}$$

$$\mathbf{f}^T = [f_1\ f_2] \quad \text{(32c)}$$

$$\mathbf{n}^T = [n_1^1\ n_1^2\ n_2^1\ n_2^2] \quad \text{(32d)}$$

Yield Criterion

Since the maximum γ is to be assessed from the total available shear strength, the reduction factor (or safety factor in LEM) **F** is unity. The undrained cohesions c_u of the two Mohr-Coulomb layers are both assumed to be unity, and lengths of the two layers are $\overline{\Delta s}_1 = 1.3454$; $\overline{\Delta s}_2 = 1.4142$. Hence, $C_1 = 1.3454$; $C_2 = 1.4142$; and from (7) we have

$$\mathbf{b}^T = [1.3454 \quad 1.4142] \qquad (33)$$

Since $\phi_u = 0$; $\mathbf{Z}$ = a 2 × 4 null matrix.

The self-weight of the soil mass will be considered as the "activating force," and the unit weight of the material is used as the load parameter λ. Therefore the body forces of element i are

$$\mathbf{p}^i = \mathbf{p}_f^i + \lambda \mathbf{p}_o^i = \mathbf{0} + \lambda \begin{bmatrix} 0 \\ V_i \\ 0 \end{bmatrix} \qquad (34)$$

where V_i = volume of element i, $i = 1, 2$; $V_1 = 0.45$; $V_2 = 0.05$ for unit thickness of the cut. For the whole system

$$\mathbf{p}_f = \mathbf{0} \qquad (35a)$$

$$\mathbf{p}_o^T = [0 \quad 0.45 \quad 0 \quad 0 \quad 0.05 \quad 0] \qquad (35b)$$

The compact primal program takes the form of LP (36)

$$\text{minimize } Z = [\mathbf{b}^T \quad -\mathbf{p}_f^T] \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{q}} \end{bmatrix}$$

$$\begin{bmatrix} \cdot & \mathbf{p}_o^T \\ \mathbf{I} & -\mathbf{A}_1 \\ -\mathbf{Z}^T & -\mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{q}} \end{bmatrix} \begin{matrix} = \\ = \\ \geq \end{matrix} \begin{bmatrix} 1 \\ \cdot \\ \cdot \end{bmatrix} \qquad (36)$$

$$\dot{\mathbf{u}} \geq \mathbf{0}; \quad \dot{\mathbf{q}} \geq \mathbf{0}$$

The compact dual program is

$$\text{maximize } W = [1 \mid \cdot \mid \cdot] \begin{bmatrix} \lambda \\ --- \\ f \\ --- \\ n \end{bmatrix}$$

$$\begin{bmatrix} \cdot & \mathbf{I} & -\mathbf{Z} \\ \mathbf{p}_o & -\mathbf{A}_1^T & -\mathbf{A}_2^T \end{bmatrix} \begin{bmatrix} \lambda \\ \mathbf{f} \\ \mathbf{n} \end{bmatrix} \begin{matrix} \leq \\ = \end{matrix} \begin{bmatrix} \mathbf{b} \\ -\mathbf{p}_f \end{bmatrix} \qquad (37)$$

$$\mathbf{f} \geq \mathbf{0}; \quad \mathbf{n} \geq \mathbf{0}$$

By introducing, where necessary, surplus, slack, or artificial variables, the LPs are converted to the standard form such that the simplex algorithm can be applied. Rough estimates of the relative computational effort can be made on the assumption that the effort varies as the product of the number of variables and the cube of the number of constraints (Munro 1979). For this example problem, since the compact primal program requires less computational effort than the compact dual program, the solution will be obtained through the former.

The optimal solution of LP (36), together with the static solution via simplex multipliers, is $\lambda = 4$; $f_1 = 1.3379$; $f_2 = 1.4142$; $n_1 = n_3 = 0$; $n_2 = 1.2041$; $n_4 = 1.4142$; $\dot{u}_1 = 0$; $\dot{u}_2 = 2.8284$; $\dot{q}_1 = \dot{q}_4 = 0$; $\dot{q}_2 = \dot{q}_5 = 2$;

and $\dot{q}_3 = \dot{q}_6 = 2$. The computed slip line is along layer 2, which intersects the free boundary at an angle of 45°. The computed critical value of the parameter $\gamma H/c_u = \lambda$ is 4, which is the reciprocal of Taylor's (1948) stability number ($\phi = 0°$) for a plane slip surface, i.e. $c_u/(\gamma H) = 0.25$. This value can also be obtained by equating the rates of internal and external work when layer 2 is considered as the slip line. This is the unique solution to the discretized soil mass and is an upper-bound solution to the continuum.

This example presents a procedure of manually calculating the matrices in LPs (24) and (25) for illustrative purposes. For practical use of the present method, a computer program has been written, which can handle both triangular and quadrilateral elements to build up automatically the dual linear program (25) [the primal program (24) can be easily obtained by transposing the matrices]. Only basic input data are required. A numerical algorithms group (NAG) library routine, which encodes the revised simplex algorithm, is utilized to solve the LP.

The simplicity of the example enables the underlying concepts to be clearly illustrated. In Chuang (1992), the present method is successfully applied to a range of problems in bearing capacity and slope stability in which the failure surfaces have various different shapes, for cohesive and frictional materials possessing homogeneous and inhomogeneous properties, external forces acting on the soil mass with varying pore pressure, and tension cracks filled with water.

CONCLUSIONS

The limit analysis of stability problems in geomechanics has been successfully formulated as a pair of primal-dual linear programs, which encode, respectively, the kinematic and static limit theorems in a discrete version, for rigid-perfectly plastic materials with a piecewise linearized yield locus and an associated flow rule. A general computer program has been developed, which makes the present method simple to implement.

For an assumed FE mesh, the solution of either linear program identifies the critical collapse mechanism among all the possible failure mechanisms contained within the given mesh, and gives the corresponding values of both static and kinematic variables, together with the critical load parameter. The profile of the critical failure surface is selected by the program automatically as part of the solution. This is the solution, which is both kinematically and statically admissible, to the discretized system; for the continuum, it is an upper-bound solution.

This method could readily give the exact solution to the stability analysis of a rock mass with multiple planes of weakness. It may also be extended to the seismic analysis of stability by including statically equivalent horizontal and vertical seismic forces.

The present method provides the solutions accurately for given values of cohesion and angle of internal friction. The shear-strength parameters of soils are usually approximately known. Sensitivity analyses of dual LP (25) can be carried out to determine the variation of the results due to small changes in the assumed strength parameters. If engineering judgments are available to be incorporated in the assessment of soil properties, fuzzy mathematical programming (Chuang and Munro 1983; Chuang et al. 1986; Munro and Chuang 1986) can be utilized for the stability analysis. Hence engineering decisions can be made based on a better understanding of the problem.

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Appendix II. Notation

The following symbols are used in this paper:

$\mathbf{A}_1^m, \mathbf{A}_2^m$ = element matrices transforming $\dot{\mathbf{q}}$ to $\Delta\dot{\mathbf{u}}$ and $\Delta\dot{\mathbf{w}}$, respectively;
$\mathbf{A}_1, \mathbf{A}_2$ = matrices transforming $\dot{\mathbf{q}}$ to $\dot{\mathbf{u}}$ and $\dot{\mathbf{w}}$, respectively;
b_l = $C_l/F - (\zeta_l^1 r_l^1 + \zeta_l^2 r_l^2)$;
C_l = integral of cohesion along layer l;
c_u = undrained cohesion;
F = reduction factor;

$\mathbf{f}$ = vector of tangential forces on Mohr-Coulomb layers;
H_i = height of element measured perpendicular to side i;
$\mathbf{I}$ = identity matrix;
L = total number of layers;
$\mathbf{n}$ = vector of normal forces on Mohr-Coulomb layers;
$\mathbf{p}$ = vector of body forces of elements;
$\mathbf{p}$ = $(\mathbf{p}_f + \lambda\, \mathbf{p}_o)$;
$\mathbf{p}_a$ = vector of activating forces;
$\mathbf{p}_a$ = $\lambda \mathbf{p}_o$;
$\mathbf{p}_f$ = vector of fixed load component of $\mathbf{p}$;
$\mathbf{p}_o$ = vector of activating forces associated with unit value of λ;
$\dot{\mathbf{q}}$ = vector of degrees of freedom of centroids of elements;
r_l^i = integral of pore pressure along half of layer l for contact node i, $i = 1, 2$;
$\mathbf{s}^+, \mathbf{s}^-$ = vectors of slack variables;
$\dot{\mathbf{u}}$ = vector of tangential deformation rates of layers;
$\dot{\mathbf{u}}$ = $\dot{\mathbf{u}}^+ - \dot{\mathbf{u}}^-$;
$\dot{\mathbf{u}}^+, \dot{\mathbf{u}}^-$ = vectors of nonnegative components of $\dot{\mathbf{u}}$;
$\dot{\mathbf{w}}$ = vector of normal deformation rates of layers;
$\mathbf{Z}$ = diag $[(\zeta_1^1, \zeta_1^2), \ldots, (\zeta_l^1, \zeta_l^2), \ldots, (\zeta_L^1, \zeta_L^2)]$;
γ = unit weight of soil;
Δs_i = length of element side i that is opposite to node i;
$\overline{\Delta s}_l$ = length of interfacing layer l;
$\Delta \dot{u}_i$ = tangential displacement rate of side i of element;
$\Delta \dot{w}_1^1, \Delta \dot{w}_1^2$ = normal displacement rates at two contact nodes along side i of element;
ζ_l^i = $(\tan \phi_l'^i)/F$;
θ_i = clockwise rotation from horizontal axis of side i;
λ = load parameter;
$\boldsymbol{\sigma}$ = vector of surplus variables; and
$\phi_l'^i$ = angle of internal friction in terms of effective stress at contact node i of layer l, $i = 1, 2$.